

# Package: UPSvarApprox (via r-universe)

August 28, 2024

**Type** Package

**Title** Approximate the Variance of the Horvitz-Thompson Total Estimator

**Version** 0.1.4

**Date** 2023-08-26

**Description** Variance approximations for the Horvitz-Thompson total estimator in Unequal Probability Sampling using only first-order inclusion probabilities. See Matei and Tillé (2005) and Haziza, Mecatti and Rao (2008) for details.

**License** GPL-3

**Encoding** UTF-8

**RoxygenNote** 7.2.3

**BugReports** <https://github.com/rhobis/UPSvarApprox/issues>

**Repository** <https://rhobis.r-universe.dev>

**RemoteUrl** <https://github.com/rhobis/upsvarapprox>

**RemoteRef** HEAD

**RemoteSha** 15e803d6136e89ff7f2f80eachb0907bfc5455fbc

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UPSvarApprox-package    *UPSvarApprox: Approximate the variance of the Horvitz-Thompson estimator*

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### Description

Variance approximations for the Horvitz-Thompson total estimator in Unequal Probability Sampling using only first-order inclusion probabilities. See Matei and Tillé (2005) and Haziza, Mecatti and Rao (2008) for details.

### Variance approximation

The package provides function `Var_approx` for the approximation of the Horvitz-Thompson variance, and function `approx_var_est` for the computation of approximate variance estimators. For both functions, different estimators are implemented, see their documentation for details.

### Author(s)

**Maintainer:** Roberto Sichera <rob.sichera@gmail.com>

### References

Matei, A.; Tillé, Y., 2005. Evaluation of variance approximations and estimators in maximum entropy sampling with unequal probability and fixed sample size. *Journal of Official Statistics* 21 (4), 543-570.

Haziza, D.; Mecatti, F.; Rao, J.N.K. 2008. Evaluation of some approximate variance estimators under the Rao-Sampford unequal probability sampling design. *Metron* LXVI (1), 91-108.

### See Also

Useful links:

- Report bugs at <https://github.com/rhobis/UPSvarApprox/issues>

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approx\_var\_est    *Approximated Variance Estimators*

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### Description

Approximated variance estimators which use only first-order inclusion probabilities

### Usage

```
approx_var_est(y, pik, method, sample = NULL, ...)
```

**Arguments**

y	numeric vector of sample observations
pik	numeric vector of first-order inclusion probabilities of length N, the population size, or n, the sample size depending on the chosen method (see Details for more information)
method	string indicating the desired approximate variance estimator. One of "Deville1", "Deville2", "Deville3", "Hajek", "Rosen", "FixedPoint", "Brewer1", "HartleyRao", "Berger", "Tille", "MateiTille1", "MateiTille2", "MateiTille3", "MateiTille4", "MateiTille5", "Brewer2", "Brewer3", "Brewer4".
sample	Either a numeric vector of length equal to the sample size with the indices of sample units, or a boolean vector of the same length of pik, indicating which units belong to the sample (TRUE if the unit is in the sample, FALSE otherwise. Only used with estimators of the third class (see Details for more information).
...	two optional parameters can be modified to control the iterative procedures in methods "MateiTille5", "Tille" and "FixedPoint": maxIter sets the maximum number of iterations to perform and eps controls the convergence error

**Details**

The choice of the estimator to be used is made through the argument method, the list of methods and their respective equations is presented below.

Matei and Tillé (2005) divides the approximated variance estimators into three classes, depending on the quantities they require:

1. First and second-order inclusion probabilities: The first class is composed of the Horvitz-Thompson estimator (Horvitz and Thompson 1952) and the Sen-Yates-Grundy estimator (Yates and Grundy 1953; Sen 1953), which are available through function varHT in package sampling;
2. Only first-order inclusion probabilities and only for sample units;
3. Only first-order inclusion probabilities, for the entire population.

Haziza, Mecatti and Rao (2008) provide a common form to express most of the estimators in class 2 and 3:

$$\widehat{var}(t_{HT}) = \sum_{i \in s} c_i e_i^2$$

where  $e_i = \frac{y_i}{\pi_i} - \hat{B}$ , with

$$\hat{B} = \frac{\sum_{i \in s} a_i (y_i / \pi_i)}{\sum_{i \in s} a_i}$$

and  $a_i$  and  $c_i$  are parameters that define the different estimators:

- method="Hajek" [Class 2]

$$c_i = \frac{n}{n-1} (1 - \pi_i); \quad a_i = c_i$$

- method="Deville2" [Class 2]

$$c_i = (1 - \pi_i) \left\{ 1 - \sum_{j \in s} \left[ \frac{1 - \pi_j}{\sum_{k \in s} (1 - \pi_k)} \right]^2 \right\}^{-1}; \quad a_i = c_i$$

- method="Deville3" [Class 2]

$$c_i = (1 - \pi_i) \left\{ 1 - \sum_{j \in s} \left[ \frac{1 - \pi_j}{\sum_{k \in s} (1 - \pi_k)} \right]^2 \right\}^{-1}; \quad a_i = 1$$

- method="Rosen" [Class 2]

$$c_i = \frac{n}{n-1} (1 - \pi_i); \quad a_i = (1 - \pi_i) \log(1 - \pi_i) / \pi_i$$

- method="Brewer1" [Class 2]

$$c_i = \frac{n}{n-1} (1 - \pi_i); \quad a_i = 1$$

- method="Brewer2" [Class 3]

$$c_i = \frac{n}{n-1} \left( 1 - \pi_i + \frac{\pi_i}{n} - n^{-2} \sum_{j \in U} \pi_j^2 \right); \quad a_i = 1$$

- method="Brewer3" [Class 3]

$$c_i = \frac{n}{n-1} \left( 1 - \pi_i - \frac{\pi_i}{n} - n^{-2} \sum_{j \in U} \pi_j^2 \right); \quad a_i = 1$$

- method="Brewer4" [Class 3]

$$c_i = \frac{n}{n-1} \left( 1 - \pi_i - \frac{\pi_i}{n-1} + n^{-1} (n-1)^{-1} \sum_{j \in U} \pi_j^2 \right); \quad a_i = 1$$

- method="Berger" [Class 3]

$$c_i = \frac{n}{n-1} (1 - \pi_i) \left[ \frac{\sum_{j \in s} (1 - \pi_j)}{\sum_{j \in U} (1 - \pi_j)} \right]; \quad a_i = c_i$$

- method="HartleyRao" [Class 3]

$$c_i = \frac{n}{n-1} \left( 1 - \pi_i - n^{-1} \sum_{j \in s} \pi_j + n^{-1} \sum_{j \in U} \pi_j^2 \right); \quad a_i = 1$$

Some additional estimators are defined in Matei and Tillé (2005):

- method="Deville1" [Class 2]

$$\widehat{var}(\hat{t}_{HT}) = \sum_{i \in s} \frac{c_i}{\pi_i^2} (y_i - y_i^*)^2$$

where

$$y_i^* = \pi_i \frac{\sum_{j \in s} c_j y_j / \pi_j}{\sum_{j \in s} c_j}$$

and  $c_i = (1 - \pi_i) \frac{n}{n-1}$

- method="Tille" [Class 3]

$$\widehat{var}(\hat{t}_{HT}) = \left( \sum_{i \in s} \omega_i \right) \sum_{i \in s} \omega_i (\tilde{y}_i - \bar{\tilde{y}}_\omega)^2 - n \sum_{i \in s} \left( \tilde{y}_i - \frac{\hat{t}_{HT}}{n} \right)^2$$

where  $\tilde{y}_i = y_i/\pi_i$ ,  $\omega_i = \pi_i/\beta_i$  and  $\bar{\tilde{y}}_\omega = \left( \sum_{i \in s} \omega_i \right)^{-1} \sum_{i \in s} \omega_i \tilde{y}_i$

The coefficients  $\beta_i$  are computed iteratively through the following procedure:

1.  $\beta_i^{(0)} = \pi_i, \forall i \in U$
2.  $\beta_i^{(2k-1)} = \frac{(n-1)\pi_i}{\beta^{(2k-2)} - \beta_i^{(2k-2)}}$
3.  $\beta_i^{2k} = \beta_i^{(2k-1)} \left( \frac{n(n-1)}{(\beta^{(2k-1)})^2 - \sum_{i \in U} (\beta_i^{(2k-1)})^2} \right)^{(1/2)}$

with  $\beta^{(k)} = \sum_{i \in U} \beta_i^k, k = 1, 2, 3, \dots$

- method="MateiTille1" [Class 3]

$$\widehat{var}(\hat{t}_{HT}) = \frac{n(N-1)}{N(n-1)} \sum_{i \in s} \frac{b_i}{\pi_i^3} (y_i - \hat{y}_i^*)^2$$

where

$$\hat{y}_i^* = \pi_i \frac{\sum_{i \in s} b_i y_i / \pi_i^2}{\sum_{i \in s} b_i / \pi_i}$$

and the coefficients  $b_i$  are computed iteratively by the algorithm:

1.  $b_i^{(0)} = \pi_i(1 - \pi_i) \frac{N}{N-1}, \forall i \in U$
2.  $b_i^{(k)} = \frac{(b_i^{(k-1)})^2}{\sum_{j \in U} b_j^{(k-1)}} + \pi_i(1 - \pi_i)$

a necessary condition for convergence is checked and, if not satisfied, the function returns an alternative solution that uses only one iteration:

$$b_i = \pi_i(1 - \pi_i) \left( \frac{N\pi_i(1 - \pi_i)}{(N-1) \sum_{j \in U} \pi_j(1 - \pi_j)} + 1 \right)$$

- method="MateiTille2" [Class 3]

$$\widehat{var}(\hat{t}_{HT}) = \frac{1}{1 - \sum_{i \in U} \frac{d_i^2}{\pi_i}} \sum_{i \in s} (1 - \pi_i) \left( \frac{y_i}{\pi_i} - \frac{\hat{t}_{HT}}{n} \right)^2$$

where

$$d_i = \frac{\pi_i(1 - \pi_i)}{\sum_{j \in U} \pi_j(1 - \pi_j)}$$

- method="MateiTille3" [Class 3]

$$\widehat{var}(\hat{t}_{HT}) = \frac{1}{1 - \sum_{i \in U} \frac{d_i^2}{\pi_i}} \sum_{i \in s} (1 - \pi_i) \left( \frac{y_i}{\pi_i} - \frac{\sum_{j \in s} (1 - \pi_j) \frac{y_j}{\pi_j}}{\sum_{j \in s} (1 - \pi_j)} \right)^2$$

where  $d_i$  is defined as in method="MateiTille2".

- method="MateiTille4" [Class 3]

$$\widehat{var}(\hat{t}_{HT}) = \frac{1}{1 - \sum_{i \in U} b_i/n^2} \sum_{i \in s} \frac{b_i}{\pi_i^3} (y_i - y_i^*)^2$$

where

$$y_i^* = \pi_i \frac{\sum_{j \in s} b_j y_j / \pi_j^2}{\sum_{j \in s} b_j / \pi_j}$$

and

$$b_i = \frac{\pi_i(1 - \pi_i)N}{N - 1}$$

- method="MateiTille5" [Class 3] This estimator is defined as in method="MateiTille4", and the  $b_i$  values are defined as in method="MateiTille1"

## Value

a scalar, the estimated variance

## References

Matei, A.; Tillé, Y., 2005. Evaluation of variance approximations and estimators in maximum entropy sampling with unequal probability and fixed sample size. *Journal of Official Statistics* 21 (4), 543-570.

Haziza, D.; Mecatti, F.; Rao, J.N.K. 2008. Evaluation of some approximate variance estimators under the Rao-Sampford unequal probability sampling design. *Metron* LXVI (1), 91-108.

## Examples

```
### Generate population data ---
N <- 500; n <- 50

set.seed(0)
x <- rgamma(500, scale=10, shape=5)
y <- abs( 2*x + 3.7*sqrt(x) * rnorm(N) )

pik <- n * x/sum(x)
s <- sample(N, n)

ys <- y[s]
piks <- pik[s]

### Estimators of class 2 ---
approx_var_est(ys, piks, method="Deville1")
```

```

approx_var_est(ys, piks, method="Deville2")
approx_var_est(ys, piks, method="Deville3")
approx_var_est(ys, piks, method="Hajek")
approx_var_est(ys, piks, method="Rosen")
approx_var_est(ys, piks, method="FixedPoint")
approx_var_est(ys, piks, method="Brewer1")

### Estimators of class 3 ---
approx_var_est(ys, pik, method="HartleyRao", sample=s)
approx_var_est(ys, pik, method="Berger", sample=s)
approx_var_est(ys, pik, method="Tille", sample=s)
approx_var_est(ys, pik, method="MateiTille1", sample=s)
approx_var_est(ys, pik, method="MateiTille2", sample=s)
approx_var_est(ys, pik, method="MateiTille3", sample=s)
approx_var_est(ys, pik, method="MateiTille4", sample=s)
approx_var_est(ys, pik, method="MateiTille5", sample=s)
approx_var_est(ys, pik, method="Brewer2", sample=s)
approx_var_est(ys, pik, method="Brewer3", sample=s)
approx_var_est(ys, pik, method="Brewer4", sample=s)

```

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Var\_approx

*Approximate the Variance of the Horvitz-Thompson estimator*


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### Description

Approximations of the Horvitz-Thompson variance for High-Entropy sampling designs. Such methods use only first-order inclusion probabilities.

### Usage

```
Var_approx(y, pik, n, method, ...)
```

### Arguments

y	numeric vector containing the values of the variable of interest for all population units
pik	numeric vector of first-order inclusion probabilities, of length equal to population size
n	a scalar indicating the sample size
method	string indicating the approximation that should be used. One of "Hajek1", "Hajek2", "HartleyRao1", "HartleyRao2", "FixedPoint".
...	two optional parameters can be modified to control the iterative procedure in method="FixedPoint": maxIter sets the maximum number of iterations and eps controls the convergence error

### Details

The variance approximations available in this function are described below, the notation used is that of Matei and Tillé (2005).

- Hájek variance approximation (method="Hajek1"):

$$\tilde{Var} = \sum_{i \in U} \frac{b_i}{\pi_i^2} (y_i - y_i^*)^2$$

where

$$y_i^* = \pi_i \frac{\sum_{j \in U} b_j y_j / \pi_j}{\sum_{j \in U} b_j}$$

and

$$b_i = \frac{\pi_i(1 - \pi_i)N}{N - 1}$$

- Starting from Hájek (1964), Brewer (2002) defined the following estimator (method="Hajek2"):

$$\tilde{Var} = \sum_{i \in U} \pi_i(1 - \pi_i) \left( \frac{y_i}{\pi_i} - \frac{\tilde{Y}}{n} \right)^2$$

where  $\tilde{Y} = \sum_{i \in U} a_i y_i$  and  $a_i = n(1 - \pi_i) / \sum_{j \in U} \pi_j(1 - \pi_j)$

- Hartley and Rao (1962) variance approximation (method="HartleyRao1"):

$$\begin{aligned} \tilde{Var} = & \sum_{i \in U} \pi_i \left( 1 - \frac{n-1}{n} \pi_i \right) \left( \frac{y_i}{\pi_i} - \frac{Y}{n} \right)^2 \\ & - \frac{n-1}{n^2} \sum_{i \in U} \left( 2\pi_i^3 - \frac{\pi_i^2}{2} \sum_{j \in U} \pi_j^2 \right) \left( \frac{y_i}{\pi_i} - \frac{Y}{n} \right)^2 \\ & + \frac{2(n-1)}{n^3} \left( \sum_{i \in U} \pi_i y_i - \frac{Y}{n} \sum_{i \in U} \pi_i^2 \right)^2 \end{aligned}$$

- Hartley and Rao (1962) provide a simplified version of the variance above (method="HartleyRao2"):

$$\tilde{Var} = \sum_{i \in U} \pi_i \left( 1 - \frac{n-1}{n} \pi_i \right) \left( \frac{y_i}{\pi_i} - \frac{Y}{n} \right)^2$$

- method="FixedPoint" computes the Fixed-Point variance approximation proposed by Deville and Tillé (2005). The variance can be expressed in the same form as in method="Hajek1", and the coefficients  $b_i$  are computed iteratively by the algorithm:

1.

$$b_i^{(0)} = \pi_i(1 - \pi_i) \frac{N}{N - 1}, \quad \forall i \in U$$

2.

$$b_i^{(k)} = \frac{(b_i^{(k-1)})^2}{\sum_{j \in U} b_j^{(k-1)}} + \pi_i(1 - \pi_i)$$



a necessary condition for convergence is checked and, if not satisfied, the function returns an alternative solution that uses only one iteration:

$$b_i = \pi_i(1 - \pi_i) \left( \frac{N\pi_i(1 - \pi_i)}{(N - 1) \sum_{j \in U} \pi_j(1 - \pi_j)} + 1 \right)$$

### Value

a scalar, the approximated variance.

### References

Matei, A.; Tillé, Y., 2005. Evaluation of variance approximations and estimators in maximum entropy sampling with unequal probability and fixed sample size. *Journal of Official Statistics* 21 (4), 543-570.

### Examples

```
N <- 500; n <- 50

set.seed(0)
x <- rgamma(n=N, scale=10, shape=5)
y <- abs( 2*x + 3.7*sqrt(x) * rnorm(N) )

pik <- n * x/sum(x)
pik1 <- outer(pik, pik, '*'); diag(pik1) <- pik

### Variance approximations ---
Var_approx(y, pik, n, method = "Hajek1")
Var_approx(y, pik, n, method = "Hajek2")
Var_approx(y, pik, n, method = "HartleyRao1")
Var_approx(y, pik, n, method = "HartleyRao2")
Var_approx(y, pik, n, method = "FixedPoint")
```

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